Lecture 5

6.5: Exponential Growth and Decay

Many things grow or decay at an exponential rate, such as radioactive substances and population.

This type of growth/decay means the quantity changes in proportion to its size.

Mathematically, this is described by:

$y'(t) = k\,y(t)$, where k is the rate of growth/decay and $y(t)$ is the quantity at time t.

If $k>0$, this is the law of natural growth, and if $k<0$, this is the law of natural decay. This is a differential equation, the solution to which is:

$y(t) = Ce^{kt}$, where C is a constant.

Given an initial value $y(0) = y_0$, we can find a specific value for C:

$$y(0) = Ce^0 = C = y_0 \Rightarrow y(t) = y_0 e^{kt}$$

Population Growth

Let $P(t)$ denote the population size at time t. Then $P'(t) = kP(t)$, where k is the <u>relative growth rate</u>, i.e., $k = \frac{P'(t)}{P(t)}$. If $P(0) = P_0$ is the initial population, then the model for the population is given by

$$P(t) = P_0 e^{kt}$$

<u>Ex</u>: In Japan, there is an island called Ōkunoshima, a.k.a. Rabbit Island. In 1971, 8 rabbits were released onto the island, and, as of 2011, there are about 300 rabbits.

- What is the relative growth rate of the rabbit population?
  $P(40) = 8e^{40k} = 300 \Rightarrow e^{40k} = 37.5 \Rightarrow k = \frac{1}{40}\ln(37.5) \approx 0.09$
- What is $P(0) = P_0$? $P(0) = P_0 = 8$
- What differential equation does $P(t)$ satisfy? $P'(t) = 0.09P(t)$
- $P(t) = 8e^{0.09t}$
- Approximate rabbit population in 2021? $P(50) = 8e^{4.5} \approx 720$

Radioactive Decay

Let $m(t)$ denote the mass at time t. Then $m'(t) = k\,m(t)$. Since the substance is decaying, $k<0$, and so the relative decay rate is

$$-k = \frac{-m'(t)}{m(t)}.$$

If $m(0) = m_0$ is the initial mass, the model for the remaining mass is:

$$m(t) = m_0 e^{kt}$$

The decay rate of radioactive substances is often described in terms of half-life, the amount of time it take for a substance to lose half of its mass. If we call the half-life of the substance: h, we can recover the familiar formula from chemistry/physics:

$$m(h) = m_0 e^{kh} = \frac{m_0}{2} \Rightarrow e^{kh} = 2^{-1} \Rightarrow kh = -\ln 2$$

$$\Rightarrow k = \frac{-\ln 2}{h} \quad \left(\text{or } h = \frac{-\ln 2}{k}\right)$$

Ex: ^{233}Pu has a half-life of 20 minutes. Suppose we have a 100mg sample of ^{233}Pu.

- Find a formula for the mass remaining after t minutes: $m_0 = 100$, $h = 20 \Rightarrow k = \frac{-\ln 2}{20} \approx -0.035$

$$\boxed{m(t) = 100e^{-0.035t}}$$

- How much remains after one hour?

$$m(60) = 100e^{-0.035 \cdot 60} = 100e^{-2.1} \approx 12.25$$

$$\boxed{\text{About } 12.25\,\text{mg}}$$

- How long will it take for there to be 10mg of ^{233}Pu remaining?

Solve for t in: $m(t) = 100e^{-0.035t} = 10$

$$\Rightarrow e^{-0.035t} = \frac{1}{10} \Rightarrow -0.035t = \ln\frac{1}{10} = -\ln 10$$

$$\Rightarrow t = \frac{\ln 10}{0.035} \approx \boxed{65.79 \text{ minutes}}$$

Compound Interest

Compounded Continuously

Suppose we invest A_0 dollars into an account with interest rate r, compounded continuously. After t years, the amount in the account is given by:

$$A(t) = A_0 e^{rt}$$

Ex: John invests \$1000 into an account paying 5% interest, compounded continuously. How much is in the account after 5 years? $r = 5\% = 0.05$, $A_0 = 1000$

$$A(5) = 1000e^{0.05 \cdot 5} = 1000e^{0.25} \approx 1284.03$$

$\boxed{\text{About } \$1284.03}$

How long until the balance is \$2000?

Solve for t in:

$$A(t) = 1000e^{0.05t} = 2000 \Rightarrow e^{0.05t} = 2$$

$$\Rightarrow t = \frac{1}{0.05} \ln 2 \approx 13.86$$

$\boxed{\text{About } 13.86 \text{ years}}$

Compounded Discretely

The alternative to compounding continuously is to compound n times a year, e.g.,

compounded	annually	semiannually	quarterly	monthly	daily	...
$n =$	1	2	4	12	365	

If the account has an interest rate of r, and is compounded n times a year, after t years an initial investment of A_0 dollars will have value

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}$$

If we let $n \to \infty$, we see the recovery of the compounded continuously case.

<u>Ex</u>: If \$50,000 is borrowed at 10% interest, compounded quarterly, how much is owed after 4 years?

$$A(4) = 50000\left(1+\frac{0.1}{4}\right)^{4\cdot 4} = 50000(1.025)^{16}$$

$$\approx 74225.28$$

$\boxed{\text{About } \$74{,}225.28}$